A finite-difference discrete-ordinate iterative method developed to solve the three-dimensional (3D) radiative transfer equation is described. The method is applicable to a volume of ocean with position dependent volumetric absorption and scattering coefficients. Input quantities include sun position and sky radiance distribution, scattering phase function, and absorbing, reflecting, or emitting objects within the ocean volume. A solution of the one-dimensional radiative transfer equation is used to provide boundary values for the 3D solutions. Radiance distributions are calculated which model recent ocean radiance measurements showing ship shadow effects. The results are compared.
1. INTRODUCTION

The finite-difference discrete-ordinate method has been applied to a variety of problems in steady state radiative transfer\(^1,2,3\). Each of the applications possessed some form of symmetry so that the three-dimensional problem (actually five dimensions consisting of three in space and two in direction) could be reduced by eliminating at least one spatial dimension. This paper describes an iterative method for solving the complete three-dimensional problem for in-water radiance. Solutions can be obtained for water regions where the volume absorption and scattering coefficients vary with position.

We compare the solution to radiance distributions measured on a sunny day from the stern of a surface ship at various depths in ocean water. The solution incorporates the experimental sun-ship orientation, measured values of absorption and scattering, a model of the ship, and specified functions for the scattering phase function and the sky radiance distribution.

2. RADIATIVE TRANSFER EQUATIONS

2.1 Three-Dimensional Model

The radiance distribution in a scattering and absorbing medium satisfies the radiative transfer equation

\[
\frac{dL(x)}{ds} + c(x)L = \int_{-\pi}^{\pi} \int_{0}^{\pi} p(\alpha L(x,\phi',\theta') \sin \theta' d\theta' d\phi' \quad (1)
\]

In this equation \( L \) is the radiance and is a function of position \( x = (x, y, z) \) in a Cartesian
INTRODUCTION

The discrete-ordinate method has a natural symmetry so that the three-dimensionally five dimensions space and two in direction) allowing at least one spatial direction. Each of the applications describes an iterative method for a three-dimensional problem for which a solution can be obtained for water absorption and scattering position.

Equation to radiance distributions from the stern of a surface in ocean water. The solution method is sun-ship orientation, absorption and scattering, a model for scattering and the radiative radiance distribution.

TRANSFER EQUATIONS

The solution in a scattering and coordinate system and direction ($\phi, \theta$) where $\theta$ is the polar zenith angle and $\phi$ is the azimuth angle. $\frac{dL}{ds}$ is the directional derivative of $L$ in the direction specified by ($\phi, \theta$). The function $c(x)$ is the attenuation coefficient, and $c(x) = a(x) + b(x)$, where $a(x)$ is the volume absorption coefficient and $b(x)$ is the volume scattering coefficient. In Equation 1 it has been assumed that any depth dependence in the volume scattering function can be factored out so that it is a product of $b(x)$ and the scattering phase function $p(\alpha)$. The scattering phase function is assumed to be a function of the angle $\alpha$ between incident and scattered radiance directions, ($\phi, \theta$) and ($\phi', \theta'$) and is normalized so that

$$\int_{-\pi}^{\pi} \int_{0}^{\pi} p(\alpha) \sin \theta' d\theta' d\phi' = 4\pi$$

The scattering angle is

$$\alpha = \cos^{-1}[\sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta']$$

The radiative transfer equation is solved in the water using a finite-difference method. A region is defined: $x_{min} \leq x \leq x_{max}$, $y_{min} \leq y \leq y_{max}$, $z_{min} \leq z \leq 0$. The radiance must be specified at the boundaries for all directions into the region. These boundary values are obtained from a solution to the radiative transfer equation in one dimension (depth) applied to the simpler problem where the radiance is uniform horizontally but still depends on depth and direction. This 1D case will be discussed later.
The finite-difference method begins by defining a grid (not necessarily uniform):

\[ \begin{align*}
  x_{\min} &= x_1 < x_2 < \ldots < x_{NX} = x_{\max} \\
  y_{\min} &= x_1 < x_2 < \ldots < x_{NY} = y_{\max} \\
  z_{\min} &= x_1 < x_2 < \ldots < x_{NZ} = z_{\max} \\
  0 &= \theta_1 < \theta_2 < \ldots < \theta_{NT} = \pi
\end{align*} \]

(4)

and for each value of \( \theta_m \) a set of \( \phi \) (possibly different for each \( \theta_m \))

\[ -\pi = \phi_{1m} < \phi_{2m} < \ldots < \phi_{NF(m)m} = \pi \]

and by defining the radiance at these grid points, \( L_{ijk\ell m} = L(x_i, y_j, z_k, \phi_\ell, \theta_m) \). The notation for \( \phi \) has the single subscript \( \ell \), with the understanding that \( \phi \) also depends upon \( m \).

The integral in Equation 1 is approximated with

\[ \int_{-\pi}^{\pi} \int_{0}^{\pi} p(\alpha) L(x_i, y_j, z_k, \phi', \theta') \sin \theta' d\theta' d\phi' \approx \]

\[ \sum_{m'=1}^{NT} \sum_{\ell'=1}^{NF(m)} \int_{-\pi}^{\pi} \int_{0}^{\pi} p(\alpha) \sin \theta' d\theta' d\phi' \]

(5)

The integration of \( p(\alpha) \) for \((\ell', m') = (\ell, m)\) is done analytically and for \((\ell', m') \neq (\ell, m)\) is approximated with a simple two-dimensional trapezoid rule with a fine \( \delta \phi', \delta \theta' \) mesh (finer than the \( \phi, \theta \) grid on which \( L \) is defined). The angular region \( A_{\ell, m} \) over which the
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\[ L(x_i', y_j', z_k', \phi', \theta') \sin \theta \sin \phi \text{d} \theta \text{d} \phi \approx \]

\[ \sum_{l,m} L_{ijl',m',} \int \int p(\alpha) \sin \theta \sin \phi \text{d} \theta \text{d} \phi \]

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\[ \text{two-dimensional trapezoid rule with a} \]

\[ \text{mesh (finer than the } \phi, \theta \text{ grid on which } L \text{)} \]

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egr integral in Equation 5 is carried out is a suitable

ion around \( \phi_{\ell', m'} \).

For no scattering the radiance \( L \) decays exponentially as a function of distance. In this case, a finite-difference approximation to the directional derivative of the radiance would have errors proportional to the spatial grid size. However, the approximation would be exact for the logarithm of the radiance. It is reasonable to assume from this that, with scattering included, the finite-difference approximation to the derivative of the logarithm of the radiance is more accurate.

With this in mind the derivative in Equation 1 is written in terms of the natural logarithm of \( L \)

\[ \frac{dL}{ds} = L \frac{d(lnL)}{ds} \]  \hspace{1cm} (6)

The finite-difference approximation is

\[ \frac{d(lnL)}{ds} \approx \frac{\ln(L_{ijk\ell m}/L_{\ast\ell m})}{\Delta r} \]  \hspace{1cm} (7)

where \( L_{\ast\ell m} \) and \( \Delta r \) are defined with the aid of Figure 1. From the point \((x_i, y_j, z_k)\) move in the direction opposite to that specified by \( \phi_{\ell}, \theta_m \) until a grid surface is reached. The distance traveled is \( \Delta r \) and the value of the radiance at the grid surface is \( L_{\ast\ell m} \). This radiance value is obtained from a geometric average of the radiance values \( L_1, L_2, L_3, L_4 \) (see Figure 1). That is, a bilinear interpolation is done using \( \ln(L_1), \ln(L_2), \ln(L_3), \ln(L_4) \) to obtain \( \ln(L_{\ast\ell m}) \).
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\[
\sum_{m'=1}^{NT} \sum_{\ell'=1}^{NF(m)} L_{ijk\ell m'} \int_{A_{\ell' m'}} p(\alpha) \sin \theta' \, d\theta' \, d\phi' \]

The integration of \( p(\alpha) \) for \((\ell', m') = (\ell, m)\) is done analytically and for \((\ell', m') \neq (\ell, m)\) is approximated with a simple two-dimensional trapezoid rule with a fine \( \delta \phi', \delta \theta' \) mesh (finer than the \( \phi, \theta \) grid on which is defined). The angular region \( A_{\ell' m'} \) over which
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\[
\int_{-\pi}^{\pi} \int_{0}^{\pi} \int_{0}^{NT} \sum_{m'=1}^{NF(m)} \sum_{\ell'=1}^{N\ell} L_{ijklm'} \int_{A_{\ell',m'}} L \sin \theta' \, d\theta' \, d\phi' \, d\phi \approx \int_{-\pi}^{\pi} \int_{0}^{\pi} \int_{0}^{NT} \sum_{m'=1}^{NF(m)} \sum_{\ell'=1}^{N\ell} L_{ijklm'} \int_{A_{\ell',m'}} L \sin \theta' \, d\theta' \, d\phi' \, d\phi \quad (5)
\]

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using \( \ln(L_1'), \ln(L_2'), \ln(L_3'), \ln(L_4') \) to obtain
\( \ln(L_{\star \ell m}) \).
Fig. 1. Finite Difference Geometry.

With this notation the finite-difference approximation to Equation 1 becomes

$$\frac{\ln(L_{ijklm}/L_{*lm})}{\Delta r} + c(z) L_{ijklm} = \frac{b(z)}{4\pi} \left[ L_{ijklm} P_{\ell m} + \text{SUM} \right]$$

(8)

where

$$P_{\ell m} = \int \int \frac{d(\alpha) \sin \theta \ d\theta \ d\phi}{A_{\ell m}}$$

(9)

and
A solution to the radiative-transfer equation

\[ \text{SUM} = \sum_{m'=1}^{NT} \sum_{\ell'=1}^{NF(m')} L_{ijk \ell' m'} \int \int p(\alpha) \sin \theta' d\theta' d\phi' \]  

(10)

For each pair of $\phi', \theta_m$ Equation 8 can easily be solved for $L_{ijk \ell m}$ in terms of previously determined or specified values of $L$. All that needs to be done is to start at the appropriate boundary and sweep through the $x_i, y_j, z_k$ grid in the correct direction. The nonlinearity introduced by the logarithm of the radiance is overcome by using Newton's method. That is, with $L^0$ a suitable initial approximation to $L_{ijk \ell m}$ calculate successive approximations for $n = 0, 1, 2, \ldots$, using

\[ L^{n+1} = \frac{L^n + \frac{b(z_k)}{4\pi} \text{SUM}}{\frac{L^{n+1}}{L_{* \ell m}} + 1} / \Delta r + c(z_k) - \frac{b(z_k)}{4\pi} P_{\ell m} \]  

(11)

If it were not for the form of the integral on the right-hand side of Equation 8 the problem would be solved. However, for each direction $(\phi, \theta_m)$ the sum requires knowing $L$ at all directions $(\phi, \theta_m)$ different from $(\phi', \theta_m)$. Thus, Equation 8 is a system of equations coupled together through $\ell$ and $m$. A straightforward iteration scheme is used to solve the equations. The integral is approximated with whatever values of $L$ are available. Zero can be used, however it is much more efficient to use the solution from the one-dimensional model discussed below. Then Equation 8 is solved to get a new $L$. This new $L$ is used to evaluate the integral, and Equation 8 is solved again, and so on until some convergence criterion is
satisfied. A reasonable convergence criterion is that for each position and direction the radiance at successive iterations must differ by less than 1%. The \( L_{ijkm} \) term in the integral is separated out from the sum in Equation 8 and included as part of the new iteration. The rest of the terms in the sum are evaluated at their most recent values. This makes the finite-difference iterative scheme fully implicit in \( L_{ijkm} \). Because \( p(\alpha) \) is a highly peaked forward scattering function for ocean water, the iteration scheme is stable and converges quickly (typically in six to ten iterations).

2.2 Object Description

The surface of an object in the three-dimensional box is described by the equation

\[
S(x,y,z) = 0
\]  

The function \( S \) can describe any surface whatsoever as long as a unique normal exists everywhere. It is a function of the continuously changing variables \( x, y, z \). That is, it is not defined at just the discrete grid points. In fact, the surface does not have to contain any of the grid points or be coincident with any grid plane. The current implementation models the surface as a Lambertian reflector with reflectivity \( r_s(x,y,z) \) specified as a function of position. The object is incorporated in the solution calculation in the following way. In moving backwards along the direction vector in Figure 1 the surface \( S \) may be encountered before a grid plane. If this is the case, then \( L_{iklm} \) is defined as
A solution to the radiative-transfer equation

\[ L_{*\ell m} = \frac{r_{S}(x, y, z)}{\pi} \int L(x', y', z', \phi', \theta') \cos \theta \sin \theta' d\theta' d\phi' \]  \hspace{1cm} (13)

where \( \cos \alpha \) is the cosine of the angle between the inward normal to the surface and the direction specified by \( \phi', \theta' \). The integration is carried out over the hemisphere where \( \cos \alpha \) is greater than zero.

2.3 One-Dimensional Model

As mentioned above, the boundary conditions for the three-dimensional model and the initial guess in the iteration scheme are obtained from the solution to a one-dimensional model\(^4\). For stratified planes, the radiance is not a function of \( x \) and \( y \) and the radiative transfer equation reduces to

\[ \cos \theta \frac{dL}{dz} + c(z)L = \frac{b(z)}{4\pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} p(\alpha)L(z, \phi', \theta') \sin \theta' d\theta' d\phi' \]  \hspace{1cm} (14)

The finite-difference equation is identical to Equation 8 with the derivative term replaced with

\[ \frac{dL}{dz} \approx L_{k\ell m} \frac{ln(L_{k\ell m}/L_{k+1\ell m})}{z_k - z_{k+1}} \]  \hspace{1cm} (15)

with \( k-1 \) chosen for radiance directed upward and \( k+1 \) for downward.
The finite-difference equation is solved exactly as for the three-dimensional case. The solution can be obtained in a few seconds instead of hours because there is only one space dimension. Only two boundary conditions need to be specified, \( L \) at \( z = z_{\text{min}} \) and \( z = 0 \).

2.4 **Boundary Conditions and Functions**

The one-dimensional equation requires only incoming radiance to be specified for the bottom and top boundaries. The only source of radiance is the sky.

At the surface \( (z = 0) \) the in-water downwelling radiance is calculated from a clear-sky radiance distribution, here taken to be a Gaussian of 10° width.

\[
L_{\text{sky}}(\alpha) = e^{-2(\alpha/10')^2} \tag{16}
\]

with \( \alpha \) the angle between the radiance direction and the peak sun radiance direction. In the water the downward radiance is the sum of the sky radiance refracted through the air-water interface and the upward radiance that is reflected at the interface. The reflected and refracted radiances are calculated according to Fresnel's law for unpolarized light. Of course the upwelling radiance that is reflected is not known beforehand and so the downwelling radiance at the surface is coupled into the iteration scheme.

At the maximum depth \( (z = z_{\text{min}}) \) incoming radiance is calculated assuming the water is a Lambertian reflector. The water reflectivity is obtained by
A solution to the radiative-transfer equation is solved exactly for the one-dimensional case. The solution can be extended instead of hours because of the dimension. Only two boundary conditions are specified, \( L(z = z_{\text{min}}) = 0 \) the in-water downwelling radiances from a clear-sky radiance taken to be a Gaussian of 10° width.

\[
0^\circ)^2
\]  

when the radiance direction and the angle. In the water the downward the sky radiance refracted interface and the upward radiance is the reflected and the interface. The reflected and calculated according to polarized light. Of course the reflected is not known downwelling radiance at the iteration scheme.

An incoming radiance at the water is a Lambertian reflectivity obtained by extrapolating the reflectivity calculated at shallower depths. As at the surface, the incoming radiance at depth depends upon the solution and is coupled into the iteration scheme.

Any scattering phase function can be incorporated into the model. The one used here is a curve fit to measurements reported by Petzold\(^5\) taken at AUTEC (Table 1). This function divided by \(4\pi\) is plotted in Figure 2.

The functions \(a(z)\) and \(b(z)\) are taken as piecewise linear functions fit to data.

### 3. Finite-difference Implementation

A critical part of the finite-difference solution method is specifying the five-dimensional grid (Equation 4). Spatial points must be close together so

<table>
<thead>
<tr>
<th>(\sigma) (degrees)</th>
<th>(p(\sigma)) ((\sigma) in radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0^\circ \leq</td>
<td>\sigma</td>
</tr>
<tr>
<td>(1^\circ \leq</td>
<td>\sigma</td>
</tr>
<tr>
<td>(2^\circ \leq</td>
<td>\sigma</td>
</tr>
<tr>
<td>(20^\circ \leq</td>
<td>\sigma</td>
</tr>
<tr>
<td>(40^\circ \leq</td>
<td>\sigma</td>
</tr>
<tr>
<td>(80^\circ \leq</td>
<td>\sigma</td>
</tr>
</tbody>
</table>
that the approximation to the directional derivative (Equation 7) is accurate. Tests with the one-dimensional model showed that points should be spaced no further apart than one scattering length, that is $\Delta x, \Delta y, \Delta z < 1/b$.

Since the radiance entering the region is taken from the solution to the one-dimensional model, the boundaries must be sufficiently far away that the influence of the inhomogeneous region on the incoming radiance is negligible. Tests using the model have shown that incoming radiance a couple of scattering
lengths away from the inhomogeneity is practically independent of the inhomogeneity.

Also the boundaries must be sufficiently far away to allow the radiance directions of interest to include the inhomogeneity at the distances of interest. For example, in the surface ship case, to "see" the hull from a depth of 100 meters at an angle of 45° the computational region would have to be at least 200 meters horizontally by 100 meters vertically.

The distribution of radiance directions must be fine enough so that the summation approximation to the integral (Equation 5) is accurate. In angular regions where the radiance is almost constant, as in the upward directed hemisphere, the points can be spaced far apart. Thirty degrees along a great circle arc is adequate. Where the radiance changes rapidly, as near the sun radiance direction, the points must be spaced only a few degrees apart as measured along a great circle arc.

The actual number of points used in a calculation is limited by computer memory and time available to obtain a solution. External storage could be used to permit larger array sizes but this has not been done. On an Alliant FX/4 using one computing element a solution with 21 x-values, 33 y-values, 25 z-values, and 142 direction values required about 2-1/2 hours CPU time. The solution time is directly proportional to the number of spatial grid points and proportional to the square of the number of directions.

4. COMPARISONS OF MEASUREMENTS AND CALCULATIONS

A new instrument was used to measure underwater spectral radiance distributions in the ocean off San
Diego, Ca. This instrument, an electro-optic radiance distribution camera system (RADS)\(^6,7\) allows a complete spectral radiance distribution to be measured in a short period of time and the reduction of these data to absolute radiometric values. The radiance data and measurements of the beam attenuation coefficient were obtained on October 14, 1988 between the hours of 12:48 and 13:28 PST at 33° 43.9' N, 118° 47.5' W. The instruments were deployed from the stern of the ship R/V R.G. Sproul.

The volumetric absorption and scattering coefficients, functions of depth, used in our modeling with the 1D and 3D models, were calculated from the data\(^8\) (Table 2). Table values were linearly interpolated at the other depths needed.

The spatial grid points in Figure 3a are used in modeling the radiance measurement experiment. The scattering length (Table 2) was about 4 meters and grid spacings of 2.5 meters along the ship direction and 2.0 meters transverse are chosen. Grid spacing in depth is 2.5 meters. The ship is modeled using several plane

### TABLE 2
Values of the Volumetric Absorption and Scattering Coefficients (a(z) and b(z)) from Radiance Distribution Data.

<table>
<thead>
<tr>
<th>Depth z (m)</th>
<th>a(z) (m(^-1))</th>
<th>b(z) (m(^-1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.8</td>
<td>0.0648</td>
<td>0.249</td>
</tr>
<tr>
<td>9.9</td>
<td>0.0553</td>
<td>0.251</td>
</tr>
<tr>
<td>44.8</td>
<td>0.0505</td>
<td>0.156</td>
</tr>
<tr>
<td>49.6</td>
<td>0.0341</td>
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</tr>
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</table>

A solution to the radiative-transfer problem

surfaces. The waterline cross-section is shown in figure 3a. The sides are vertical and the bottom has a shallow V shape with keel at 2.4 meters depth. The entire hull is assumed to have a reflectivity of 10%. The location of the camera system is indicated by the spot (W) off the stern. Sun radiance was from the left along the axis of the ship.

The angular grid system is shown in Figure 3b. The radius in the upwelling display represents zenith angles from 0° to 90°, while in the downwelling display it represents angles from 180° to 90°. The upwelling angles are widely spread because the radiance distribution is quite smooth there. The downwelling angles are chosen more closely spaced because the radiance varies more rapidly at those angles. In addition, angles are clustered closely around the direction of the sun radiance in water, \(\theta = 150°\) and \(\phi = 0°\).

Figures 4 and 5 show contour levels of the radiance angular distribution for both measurements and 3D calculations at 25 and 55 meters depth. The zenith angles are proportional to radial distance as in Figure 3b. The azimuth angle is zero at the far right of each plot with \(\phi\) increasing clockwise for upwelling and counter-clockwise for downwelling. The upwelling and downwelling parts of a distribution will correspond, edge to edge, if they are folded together along a horizontal axis. Contour intervals are 2.5 dB beginning from the radiance maximum on the downwelling plots and beginning from the radiance minimum on the upwelling plots. Values for the minimum and maximum can be obtained from Figure 7.
3a. Horizontal Plane Grid Points

3b. Angular Grid Points

Fig. 3. Angular and Horizontal Plane Grid Points Used in Modeling the Ship-Shadowing Measurements.
Fig. 4. Contour Levels of Measurements and 3D Calculations of Downwelling and Upwelling Radiance. 25 meters depth.
Fig. 5. Contour Levels of Measurements and 3D Calculations of Downwelling and Upwelling Radiance. 55 meters depth.
The measurements clearly show a reduction in radiance from the direction of the ship. Also evident are effects of the instrument support system. A triangular arrangement of three cables held the platform on which RADS was mounted. The cables show up as line like features in the contour plots which point slightly off center because the camera was not in the center but was near one of the cables. The ship effect is stronger at 25 meters, but is still evident at 55 meters.

The calculations show the ship effects which are similar to the measurement. The cables are not included in the model and so do not appear.

Plots of the calculations of radiance values in the sun plane (\(\phi = 0\), i.e. along the dotted lines in Figures 4 and 5) are given in Figure 6 for depths ranging from 25 meters to 55 meters. The abscissa is the zenith angle chosen positive toward \(\phi = 0\), so that the sun radiance is at 150° and downwelling radiance lies between 90° and 270°. The effect of the ship's shadow can be seen here as the difference between the 1D and the 3D calculations. The locations \((x, y, z)\) for which these plots have been calculated are in the direct rays of the sun. Radiance arriving from the vicinity of the ship can be seen to be reduced by a factor of two at 25 meters and by 20% at 55 meters.

The sun-plane plots of the 3D calculations and the measurements are compared in Figure 7 for depths at 25 and 55 meters. The measurements have been scaled so that the peak value for 25 meters depth has the same value as the peak in the calculation at the same depth. At 25 meters the calculated profile is a little
Fig. 6  1D and 3D Calculations—Radiance Distributions in Sun-Plane Zenith Angle, at Various Depths.

narrower than the measured one. In the ship region the calculation gives a smooth curve that goes through the rough curve given by the data. At 55 meters the profiles are still quite close. The data peak value is a little larger than the model peak value. The data
Calculations: Radiance in Sun-Plane Zenith Angle, at 1.5s.

One. In the ship region the 3rd curve that 'goes through the data. At 55 meters the close. The data peak value is model peak value. The data values drop suddenly at 250° whereas the model values drop smoothly.

5. DISCUSSION AND CONCLUSIONS
A method has been described for solving the steady radiative transfer equation in three space dimensions
and all radiance directions. Realistic sky models and water optical properties can easily be incorporated. Objects can be included in the volume and can have very general shapes and reflective properties.

Model comparisons with data taken in the vicinity of a ship show good results. The perturbation in the radiance distribution caused by the ship is modeled well. The abrupt changes in the measurement around 250° from the vertical are probably due to the camera support system and not to the ship shadow. At 55 meters depth the ship position to camera position occupies angles from 180° to 216°.

In general the method of solution appears to work well. The approach used to validate the solution is to compare model results with experimental results. A theoretical assessment of the finite-difference approximation to the directional derivative and the quadrature formula used to approximate the integral needs to be done.

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